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On the structure of ultraviolet divergences in the vacuum region of quantum electrodynamics

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Abstract. Starting from a finite QED in the Johnson–Baker–Willey formulation, structures of UV divergences of $\langle 0|S|0\rangle$ in QED are analysed. To obtain the finite QED in the vacuum region several eigenvalue conditions for the bare electron charge are introduced.

1. Introduction

It is well known that the usual perturbation approach to quantum electrodynamics (QED) leads to ultraviolet (UV) divergences when calculating the S-matrix physical processes. This is true as well for the S-matrix vacuum expectation value (VEV). Although these divergences can be removed from QED by the renormalisation procedure, determining their real origin is of fundamental significance not only for QED, but also for quantum field theory. Even before applying the renormalisation procedure, there have been many attempts at obtaining the S-matrix elements without the UV divergences. One of the most attractive approaches to finite spinor QED is that of Johnson, Baker and Willey (1964, Johnson and Baker 1973 and references therein), which is further discussed by Adler (1972). Their basic idea is that the usual perturbation treatment of QED is not mathematically adequate and that one must look for a convenient modification of the perturbation approach. Within their programme, the electron propagator, the photon propagator and the vertex function become finite. However, two conditions must be imposed: (1) the electron mass is completely dynamic in origin and (2) the bare fine-structure constant should obey the definite eigenvalue condition, i.e. $f(\alpha_0)=0$.

Extrapolation of the Johnson–Baker–Willey (JBW) approach to other quantum field model theories has not yet yielded the desired results. The scalar QED in the Klein–Gordon formalism has been investigated (Fry 1973) more than any other model. In scalar electrodynamics, three eigenvalue conditions for asymptotic coupling α_0 have to be introduced and Fry's conclusion is that 'a completely finite, closed theory of scalar electrodynamics is probably internally inconsistent'. In this paper we analyse the necessary conditions for the possible existence of a finite spinor QED in the vacuum region. Although the VEV of the S-matrix is not relevant to the physical meaning of QED, it is still important to know its structure (particularly its UV divergences) in the modified perturbation approach. This can be used as a test of the applicability of the JBW programme to the larger region of electromagnetic processes (although purely theoretical).

With the help of the functional derivation method, we shall first derive two general formulae which express the exponent of $\langle 0|S|0\rangle$ as the integral over the vacuum polarisation effects (13) or the electron self-energy effects (14). Then, for finiteness, some conditions on the coupling constant are introduced for both cases: (1) the physical electron mass $m = 0$ and (2) the physical electron mass $m \neq 0$.

2. Derivation of general formulae

The vacuum expectation value of the S-matrix is defined as the vacuum–vacuum S-matrix element. We shall denote it by

$$\langle 0|S|0\rangle = \langle S\rangle_0 = S_0.$$

The VEV of the S-matrix of QED in the presence of an external current $J^\mu(x)$ is

$$S_0(J) = \left\langle T \exp\left(\mathrm{i}\int (:e_0 j^\mu(x) A_\mu(x): + J^\mu(x) A_\mu(x))\,\mathrm{d}x\right)\right\rangle_0 \tag{1}$$

where $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ is the current of the electron, T and : : are chronological and normal products respectively, and $\mathrm{d}x \equiv \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$. We shall look for a connection between S_0 and the electron and the photon Green functions. These have already been investigated in the JBW programme. Note that the S-matrix element introduced in (1) is a function of a bare charge e_0 and functional of the external current $J^\mu(x)$. Differentiation of (1) with respect to e_0 gives

$$\frac{\partial S_0(J)}{\partial e_0} = \mathrm{i}\int \mathrm{d}x \langle T(j^\mu(x) A_\mu(x) S(J))\rangle_0. \tag{2}$$

We write the following expression:

$$\langle T(j^\mu(x) A_\mu(x) S(J))\rangle_0 = -\mathrm{Tr}\,\gamma^\mu \langle T(\psi(x)\bar{\psi}(x) A_\mu(x) S(J))\rangle_0. \tag{3}$$

On the other hand, after functional derivation of the electron Green function

$$G(x, y|J) = (\mathrm{i}/S_0(J))\langle T(\psi(x)\bar{\psi}(y) S(J))\rangle_0 \tag{4}$$

we have

$$\frac{\delta G(x, y|J)}{\delta J^\mu(z)} = \frac{\mathrm{i}^2}{S_0(J)}\langle T(\psi(x)\bar{\psi}(y) A_\mu(z) S(J))\rangle_0 - \mathrm{i}G(x, y|J) U_\mu(z|J) \tag{5}$$

where

$$U_\mu(z|J) = (1/S_0(J))\langle T(A_\mu(z) S(J))\rangle_0.$$

The expression (2) with the use of (3) and (5) can now be written as

$$\frac{1}{S_0}\frac{\partial S_0(J)}{\partial e_0} = \mathrm{i}\int \mathrm{d}x\, \mathrm{Tr}\left(\gamma^\mu \frac{\delta G(x, x|J)}{\delta J^\mu(x)}\right) - \int \mathrm{d}x\, \mathrm{Tr}(\gamma^\mu G(x, x|J) U_\mu(x|J)). \tag{6}$$

To transform further the first term on the right-hand side in (6), we shall introduce derivation by U_μ. Using the limit $U_\mu \to 0$ ($J_\mu \to 0$), we obtain

$$\frac{\delta}{\delta J^\mu(x)} = \int \mathrm{d}z \frac{\delta U_\nu(z|J)}{\delta J^\mu(x)} \frac{\delta}{\delta U_\nu(z|J)} = -\int \mathrm{d}z\, D_{\mu\nu}(x, z|J) \frac{\delta}{\delta U_\mu(z|J)}, \tag{7}$$

$$\frac{\delta G(x, y|U)}{\delta U_\mu(z)} = e_0 \int \mathrm{d}x'\,\mathrm{d}y'\, G(x, x'|U)\Gamma^\mu(x', y', z)G(y', y|U), \tag{8}$$

where

$$\Gamma^\mu(x, y, z) = -\delta G^{-1}(x, y)/e_0\delta U_\mu(z).$$

Inserting (7) and (8) in (6) and accounting for the limit $U_\mu \to 0$ completely, we obtain

$$\frac{1}{S_0}\frac{\partial S_0}{\partial e_0} = -\mathrm{i}e_0 \int \mathrm{d}x\,\mathrm{d}y\, D_{\mu\nu}(x, y)\,\mathrm{Tr}(\gamma^\mu G(x, x')\Gamma^\nu(x', y', y)G(y', x))\,\mathrm{d}x'\,\mathrm{d}y',$$

$$S_0(e_0 = 0) = 1. \tag{9}$$

If we define L by $L = -\mathrm{i}\ln S_0$, then the last equation can be written in either of the following forms:

$$\frac{\partial L}{\partial e_0} = \frac{\mathrm{i}}{e_0}\int \mathrm{d}x\,\mathrm{d}y\, D_{\mu\nu}(x, y)\pi^{\mu\nu}(x, y), \qquad L(e_0 = 0) = 0, \tag{10}$$

$$\frac{\partial L}{\partial e_0} = \frac{\mathrm{i}}{e_0}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{Tr}(\Sigma(x, y)G(y, x)), \qquad L(e_0 = 0) = 0. \tag{11}$$

Here, we have introduced the vacuum polarisation tensor $\pi^{\mu\nu}(x, y)$ and the electron self-energy $\Sigma(x, y)$. Equation (10) expresses the S-matrix VEV through the vacuum polarisation effects, and equation (11) expresses the same VEV, but through the electron self-energy effects.

The result which is obtained analytically in (10) and (11) can be found using the diagram technique. The relevant Feynman diagrams are shown in figure 1. Firstly, it is important to see which diagrams correspond to the quantity L in the expansion $\langle 0|S|0\rangle = \exp(\mathrm{i}L)$. Factors e_0^n are introduced for convenience. An interplay of diagrams of $D_{\mu\nu}(x, y)$ and $\pi^{\mu\nu}(x, y)$ or $G(x, y)$ and $\Sigma(x, y)$, the following equations can be written:

$$\begin{aligned} -\mathrm{i}2nL_{(n)} &= \sum_{k=0}^{n-1}\int \mathrm{d}x\,\mathrm{d}y\, D_{\mu\nu}^{(k)}(x, y)\pi^{\mu\nu}_{(n-k)}(x, y), \\ -\mathrm{i}2nL_{(n)} &= \sum_{k=0}^{n-1}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{Tr}(\Sigma^{(k)}(x, y)G_{(n-k)}(x, y)), \\ (n &= 1, 2, 3, \ldots). \end{aligned} \tag{12}$$

Applying the operator $e_0\,\partial/\partial e_0$ to the last equations, summing over n, and using the double summation, we obtain equivalent equations, which are expressions (10) and (11).

We shall rewrite equations (10) and (11) in the momentum representation, using the corresponding Fourier transforms given by

$$D_{\mu\nu}(x, y) = \frac{1}{(2\pi)^4}\int \mathrm{d}q\, \mathrm{e}^{-\mathrm{i}q(x-y)}D_{\mu\nu}(q), \qquad \pi^{\mu\nu}(x, y) = \frac{1}{(2\pi)^4}\int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k(x-y)}\pi^{\mu\nu}(k).$$

After some standard calculations we obtain

$$\frac{\partial L}{\partial e_0} = \frac{\mathrm{i}}{e_0(2\pi)^4}\int \mathrm{d}x \int \mathrm{d}q\, D_{\mu\nu}(q)\pi^{\mu\nu}(q)$$

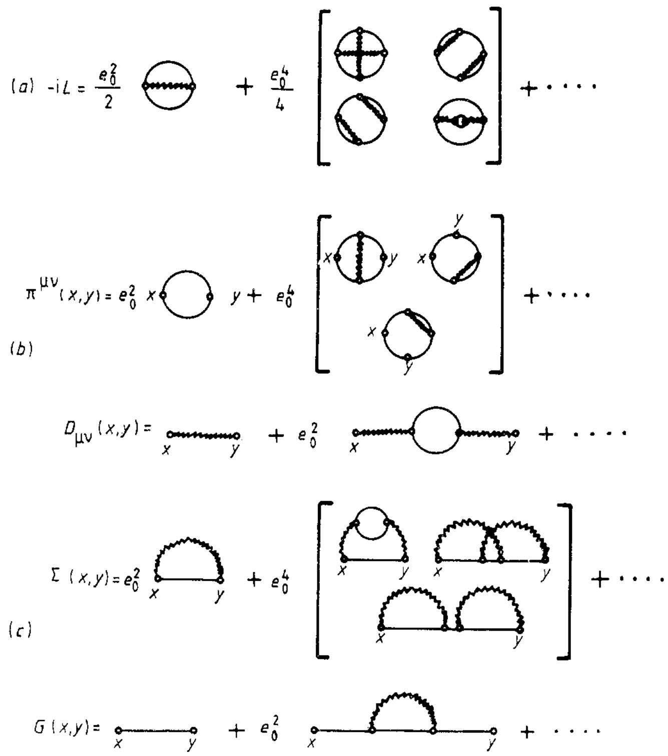

Figure 1. (*a*) Diagram representation of expansion $-\mathrm{i}L = \frac{1}{2}e_0^2L_{(1)} + \frac{1}{4}e_0^4L_{(2)} + \ldots$. One diagram in $L_{(2)}$ is taken twice in order to get the multiplying factor $e_0^4/4$. (*b*) Diagrams for vacuum polarisation quantities $\pi^{\mu\nu}(x, y)$ and $D_{\mu\nu}(x, y)$. (*c*) Graphical expansion of electron self-energy $\Sigma(x, y)$ and electron propagator $G(x, y)$.

where

$$D_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)D(q^2) - \frac{d_l}{q^2}\frac{q_\mu q_\nu}{q^2}, \qquad \pi^{\mu\nu}(q) = (g^{\mu\nu} - q^\mu q^\nu/q^2)\pi(q^2).$$

Now we have

$$\frac{\partial L}{\partial e_0} = \frac{3\mathrm{i}Vt}{e_0(2\pi)^4}\int \mathrm{d}q\, D(q^2)\pi(q^2).$$

Finally, according to the relation

$$D(q^2) = \frac{1}{\pi(q^2) - q^2} = \frac{1}{\pi(q^2) + D_0^{-1}(q^2)}$$

we obtain

$$L(e_0) = \frac{-3\mathrm{i}Vt}{(2\pi)^4}\int_0^{e_0}\frac{\mathrm{d}u}{u}\int \mathrm{d}q\left(\frac{D(q^2, u)}{D_0(q^2)} - 1\right) \tag{13}$$

or by analogy

$$L(e_0) = \frac{-\mathrm{i}Vt}{(2\pi)^4}\int_0^{e_0}\frac{\mathrm{d}u}{u}\int \mathrm{d}q\ \mathrm{Tr}\left(\frac{G(q, u)}{G_0(q)} - 1\right). \tag{14}$$

We have obtained formulae which express relations between the quantity L, the characterising VEV of the S-matrix and quantities D or G. These are known from calculations of real processes.

3. Discussion

In relation to expressions (13) and (14), according to the JBW programme and their possible future uses, there are two important points. First, the electric charge is a positive solution to the definite equation $f(e_0)=0$. Second, the bare electron mass must vanish, i.e. $m_0=0$. We can identify the upper limit in (13) and (14) with e_0, which is a solution to the JBW eigenvalue condition. Now we have finite integrands only at the end-points of the charge integration interval $0 \leqslant u \leqslant e_0$, because the vacuum polarisation function $\pi(q^2, u)$ (Johnson and Baker 1973) is

$$\pi(q^2, u)=\pi_c(q^2, u)+q^2 f(u)\int_{q^2}^{\infty}\frac{dp^2}{p^2},$$

where $\pi_c(q^2, u)$ is a finite part and $f(u)\neq 0$, except for $u=0$ and $u=e_0$. We initially introduced infinity into the exact photon propagator when we differentiated with respect to electric charge. To clarify equations (13) and (14) in relation to their finite integrands, we must redefine these integrands. This can be done by maintaining only the finite part $\pi_c(q^2, u)$ of the infinite vacuum polarisation function $\pi(q^2, u)$. Hence, for future practical use, integrands in (13) and (14) can be treated as finite functions of the electric charge u for all points of the interval $0 \leqslant u \leqslant e_0$, as well as finite functions of momentum q^2.

The infinities which still can occur, in expressions (13) and (14), are consequences of the momentum integration in the asymptotic space-like region. Since the bare electron mass must vanish ($m_0=0$), the parameter μ can be created with the dimension of mass. We distinguish three cases (Dragović *et al* 1978): (a) $\mu=0$, (b) $\mu\neq 0$, $m=0$ and (c) $\mu\neq 0$, $m\neq 0$, where m is the physical electron mass. Case (a) corresponds to the trivial solution of the Schwinger–Dyson equation for the electron propagator where, because of scale invariance, the exact electron propagator is proportional to the free propagator. It can be finite only when

$$\int_0^{e_0} A_0(u)\frac{du}{u}=0$$

where $A_0(u)$ is defined as

$$D(q^2, u)D_0^{-1}(q^2)-1=A_0(u).$$

Let us introduce the function for cases (b) and (c):

$$F(\mu^2/q^2, u)=D(q^2, \mu^2, u)D_0^{-1}(q^2)-1.$$

This is the integrand in (13). After integration over momentum q we obtain

$$\frac{1}{\mu^4}\int F\left(\frac{\mu^2}{q^2}, u\right)dq=G(u)+\lim_{q^2\to-\infty} H\left(\frac{\mu^2}{q^2}, u\right)$$

where $G(u)$ is a finite part and H is an infinite part having different divergences up to fourth order. Several eigenvalue conditions for the electric charge should be introduced to eliminate this infinite part. The question arises as to the intrinsic consistency of these conditions, including the JBW $f(e_0)$ condition. We regard this as an open question.

When the photon or electron propagators are known to a good approximation, equations (13) and (14) could be of great interest for practical calculations. An analysis of eigenvalue conditions obtained in this way could provide more insight into the nature of electric charge. We had previously accounted for the asymptotic behaviour of the solution to the Schwinger–Dyson equation for the electron propagator in the first JBW approximation (Dragović *et al* 1978). We conclude that the only bare electric charge for which the divergent part vanishes is $e_0 = 0$. Perhaps higher-order approximations will give more reasonable results.

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